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# BUCKLING STRENGTH OF FLEXIBLE CYLINDERS WITH NONUNIFORM ELASTIC SUPPORT

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**Abstract**—The critical hoop thrust which destabilizes buried flexible pipes and culverts can be an important consideration when designing structures of this type. Often, the structure is placed in a small zone of good quality backfill which is in turn surrounded by another poorer quality soil. Critical hoop thrust is studied for flexible cylinders with nonuniform solid support.

A closed form linear buckling solution is described, in addition to finite element results. The solutions presented can be used to assess the elastic stability of circular structures in square, circular and rectangular zones of elastic solid. The effects of modulus and the size of the envelope of higher modulus solid, as well as burial depth are examined. The behaviour of cylinders is also considered for a "Gibson" material which has modulus variation with depth, and for elliptical cylinders.

## INTRODUCTION

Flexible structures are often buried for use as pipelines and culverts under road and railway embankments. Earth loads and line loads acting on these structures can lead to significant compressive hoop thrusts around the pipe circumference. These compressive hoop thrusts can lead to buckling instability.

Reviews of experimental data and theoretical predictions of critical hoop thrust (Gumbel, 1983; Moore, 1989) have established that linear buckling solutions employing an elastic continuum solution to model the surrounding soil can provide useful measures of buckling strength. The theoretical solutions available feature deep or shallow burial of a circular cylindrical shell embedded in a uniform isotropic elastic continuum (Forrestal and Herrmann, 1965; Moore and Booker, 1985a, b; Moore, 1987).

In practice, however, the backfill soil placed directly adjacent to the structure is specially selected and of better quality than the soil beyond this zone. A suitable method is needed for determining the influence on buckling strength of the quality and quantity of the specially selected backfill soil and the quality of the soil beyond it.

Figure 1 shows the geometry of the problem examined in this paper. A flexible structure is buried in a zone of high modulus material (select backfill), which is in turn surrounded by a zone of poorer quality material. This situation is examined for two conditions, that of deep burial within an elastic solid, Fig. 1a, and secondly for a structure located close to a free horizontal surface (the ground surface for a shallow buried pipe), Fig. 1b.

The study commences with the presentation of a closed form solution for buried circular pipe buckling for the simplified case of a circular envelope of elastic solid, Fig. 1c. A parametric study of the problem based on that solution is then described where factors

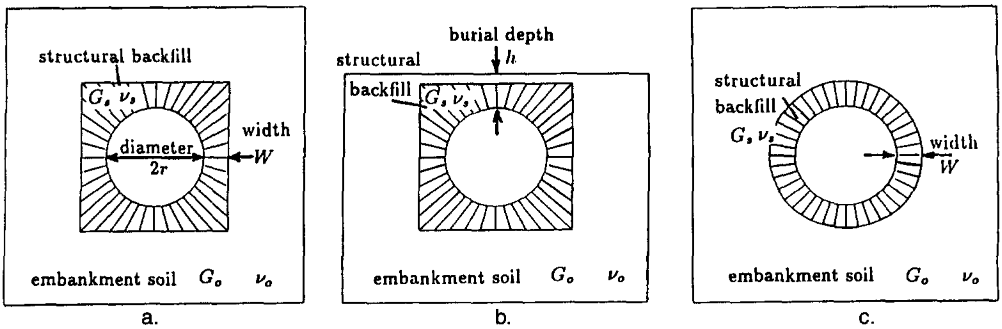


Fig. 1. Geometry of the buried pipe problem. a. Deeply buried pipe in rectangular envelope. b. Shallow buried pipe. c. Deeply buried pipe in circular envelope.

such as the width of the envelope of stiffer material and the modulus of that stiffer material and the surrounding material are examined.

Finite element analysis (Moore, 1987), is then used to examine the buckling strength of structures with more general geometry. Structures of both circular and elliptical shape are considered in envelopes of high modulus solid of various geometries.

The solutions presented will be used in a separate study to examine the stability of flexible metal culverts. After validation using laboratory or field data, this material may be used to extend the general design approach previously reported by Moore *et al.* (1988) for these structures. The solutions may also have other applications to cylinders with elastic support.

## DEEPLY BURIED CIRCULAR CYLINDERS

### *Problem description*

Figure 1c shows a circular cylinder of radius $r$ and flexural stiffness $EI$ buried in an envelope of elastic solid. Typical backfill envelopes for buried pipes are approximately rectangular in shape. For design purposes a rectangular envelope can be conservatively modelled using a ring of the select backfill. For select backfill with modulus higher than the surrounding soil, the ring width $W$ is chosen to be the minimum distance from the pipe to the lower stiffness material. For select backfill with lower modulus than the soil placed beyond it, ring width $W$ is chosen to be the maximum distance from pipe to that stiffer material beyond.

Previous studies indicate that if burial depth $h$ exceeds the pipe diameter $2r$ it is also reasonable to neglect the proximity of the ground surface (Moore, 1987). For that case, the material outside the select backfill is modelled with very distant boundaries. The select material placed adjacent to the pipe is assigned elastic shear modulus $G_s$ and Poisson's ratio $\nu_s$. The extensive zone of elastic material beyond that is assigned elastic shear modulus $G_0$ and Poisson's ratio $\nu_0$, Fig. 1c.

### *Loading*

The assessment of structural stability is often undertaken in two steps. Firstly, static analysis of the system is used to determine stress resultants which develop in the structure at working loads. Secondly, geometrically or materially nonlinear analysis is used to determine the stress resultants which lead to structural instability. Stability is maintained provided the stress resultants at working loads are less than those that cause collapse.

For the buried cylinder problem, hoop thrust develops around the pipe circumference as a result of earth loads and live loads. The present paper focuses on the thrusts which cause elastic instability. It is assumed that existing closed form and numerical procedures can be used to determine the actual thrusts which develop [Hoeg (1968) examined buried circular cylinders, Moore (1988b) considered buried elliptical cylinders, and Katona *et al.* (1976) examined shallow buried structures of various shapes].

*Uniform thrust solutions*

While the thrusts which develop around the circumference of a buried cylinder are generally nonuniform, it has been established that buckling solutions based on uniformly distributed hoop thrust provide reasonable, conservative measures of structural stability [since one wavelength of the critical deformation mode usually lies within the region of maximum thrust (Moore and Booker, 1985b; Moore, 1989)].

For the sake of simplicity, this study presents solutions for uniformly distributed thrust. To use these solutions for stability assessment of real structures, the critical thrust for a uniformly stressed cylinder should be compared with the maximum thrust that develops in the real structure at working loads.

*Closed form solution*

Linear buckling theory is used to characterize the elastic stability of the solid–structure system. Following previous studies (Moore and Booker, 1985a, b) the radial and circumferential pipe deformations $w$ and $v$ and the radial and tangential tractions at the solid–structure interface $\sigma$ and $\tau$ are represented as harmonic functions of $\theta$, the angular position on the pipe circumference, Fig. 1c,

$$w = \sum_{n=0,2,3,\ldots} w_n \cos n\theta$$

$$v = \sum_{n=2,3,\ldots} v_n \sin n\theta \tag{1}$$

$$\sigma = \sum_{n=0,2,3,\ldots} \sigma_n \cos n\theta$$

$$\tau = \sum_{n=2,3,\ldots} \tau_n \sin n\theta. \tag{2}$$

Using the shell theory of Herrmann and Armenakas (1962), stiffness relationships can be developed which relate the structural deformations to the interface tractions for harmonic $n$ and uniform thrust $N$. Linear elastic continuum mechanics, for example Timoshenko and Goodier (1970), can be used to determine stiffness equations which relate the interface deformations with the interface tractions for the nonuniform elastic continuum surrounding the cylinder. Considering equilibrium and compatibility at the interface, the uniform compressive thrust which causes elastic instability can be evaluated as

$$N = l\left(\frac{EI(n^2-1)}{r^2} + 2G_s r\chi\right). \tag{3}$$

This expression depends on the cylinder modulus $E$ and second moment of area of the cylinder $I$, the shear modulus of the soil $G_s$, and two functions of $n$, namely $l$ and $\chi$.

The variable $l$ quantifies the effect of "load behaviour" on the stability of the cylindrical shell, as discussed in detail by Moore and Booker (1985a). If the tractions $\sigma$ and $\tau$ at each point $A$ on the solid–structure interface do not rotate with the shell surface as it deforms (Fig. 2a), the behaviour is referred to as "constant directional" and

$$l = \frac{n^2}{n^2-1}. \tag{4}$$

Alternatively the tractions $\sigma$ and $\tau$ may rotate with the interface to remain normal and

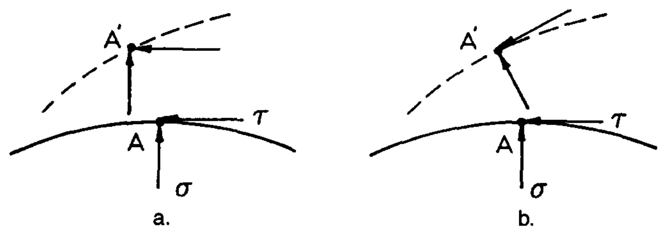


Fig. 2. Load behaviour at the solid–structure interface during buckling deformation. a. Direction of tractions is constant relative to fixed axes. b. Tractions rotate to remain normal and tangential to interface.

tangential to the cylinder surface (Fig. 2b), for which

$$l = 1. \tag{5}$$

Previous studies reveal that this load behaviour is not significant once there is a reasonable level of solid support (Moore and Booker, 1985a). All results presented in this paper are for the specific case of "constant directional" load behaviour [eqn (4)].

The variable $\chi$ works with $G_s$ in eqn (3) to quantify the level of solid resistance to structural deformations. Flexibility relationships have been developed elsewhere between boundary tractions and displacements for an annulus of elastic material (Moore, 1985). These have been used to determine the restraint provided to a flexible cylinder located at the centre of the nonuniform elastic solid shown in Fig. 1c.

For the case of a smooth pipe, there is compatibility of radial displacement only and zero shear stress at the solid–structure interface. Analysis reveals that for smooth interface

$$\chi = \frac{2}{(n^2-1)(S_1+2S_2+S_3)}. \tag{6}$$

For the case of a rough pipe where the pipe is bonded to the solid at the interface, there is full compatibility of radial and circumferential displacements and full transfer of shear stress across the interface. In that case

$$\chi = \frac{(n+1)^2S_1-2(n^2-1)S_2+(n-1)^2S_3}{2n^2(n^2-1)(S_1S_3-S_2^2)}. \tag{7}$$

The quantities $S_1$, $S_2$ and $S_3$ are defined as follows:

$$\mu = 1+\frac{W}{r} \tag{8}$$

$$A = \frac{\mu^{2+2n}-1}{2(\mu^2-1)} \tag{9}$$

$$B = \frac{\mu^{2-2n}-1}{2(\mu^2-1)} \tag{10}$$

$$C = \left(AB+\frac{n^2-1}{4}\right)(\mu^2-1) \tag{11}$$

$$D_1 = \frac{2(1-\nu_s)B+C}{(1+n)C} \tag{12}$$

$$D_2 = \frac{2(1-\nu_s)A+C}{(1-n)C} \tag{13}$$

$$D_3 = \frac{2(1-\nu_s)B - C\mu^{-2n-2}\left(1-\frac{G_s}{G_o}\right)}{C(1+n)} \tag{14}$$

$$D_4 = \frac{2(1-\nu_s)A - C\mu^{2n-2}\left(1+\frac{G_s}{G_o}(3-4\nu_o)\right)}{C(1-n)} \tag{15}$$

$$X_1 = \frac{2(1-\nu_s)B}{(1+n)C} \tag{16}$$

$$X_2 = \frac{2(1-\nu_s)A}{(1-n)C} \tag{17}$$

$$X_3 = \frac{\nu_s-1}{C} \tag{18}$$

$$S_1 = D_1 + (2X_1X_3^2 - D_4X_1^2 - D_3X_3^2)/(D_3D_4 - X_3^2) \tag{19}$$

$$S_2 = X_3 + X_3(X_3^2 + X_1X_2 - X_1D_4 - X_2D_3)/(D_3D_4 - X_3^2) \tag{20}$$

$$S_3 = D_2 + (2X_2X_3^2 - D_3X_2^2 - D_4X_3^2)/(D_3D_4 - X_3^2). \tag{21}$$

After substitution of (6) or (7), eqn (3) must be minimized with respect to $n$ to determine the critical harmonic $n_{crit}$ and the lowest and most critical thrust $N_{crit}$. Critical thrust $N_{crit}$ can be compared with expected thrust in the structure to estimate stability. The critical harmonic $n_{crit}$ can be used to estimate the likely buckling wavelength, $\lambda = 2\pi r/n_{crit}$.

For the case where $\mu = \infty$ or $G_s = G_o$ and $\nu_s = \nu_o$, then the two expressions for $\chi$ revert to the solutions for uniform homogeneous continuum (Moore and Booker, 1985a)

$$\chi = \frac{2n(1-\nu_s)-(1-2\nu_s)}{n^2(3-4\nu_s)} \tag{22}$$

for bonded interface and

$$\chi = \frac{1}{2n(1-\nu_s)+(1-2\nu_s)} \tag{23}$$

when the interface is perfectly smooth. The accuracy of the solution is further verified later in the report through comparisons with finite element results.

*Introduction to parametric study*

The closed form solution can be used to examine the influence of the size of the ring of stiff material and the stiffness of the surrounding material. It is convenient to normalize critical thrust $N_{crit}$ for pipes in an envelope of stiff material using critical thrust $N^{\infty}_{crit}$ for pipes deeply buried in an infinitely large envelope of that material,

$$N_{crit} = N^{\infty}_{crit}R_h. \tag{24}$$

Solutions for $N^{\infty}_{crit}$ have been reported elsewhere (Forrestal and Herrmann, 1965; Moore

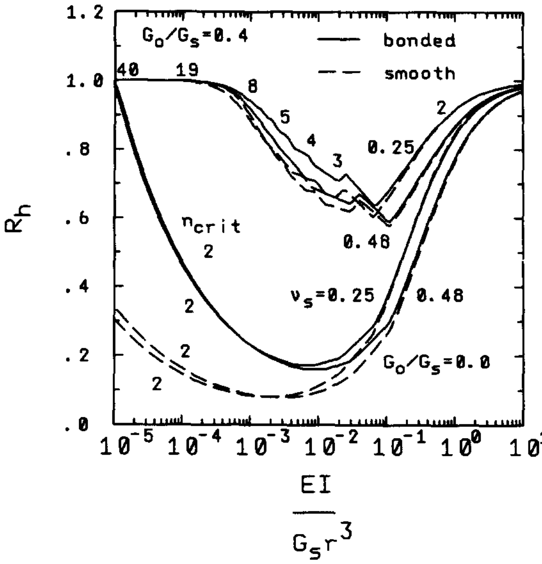


Fig. 3. Correction factor $R_h$ for pipe in circular envelope—effect of Poisson's ratio and interface condition.

and Booker, 1985a,b), and the factor $R_h$ directly quantifies the effect of the geometry of the zone of higher modulus solid and the difference in elastic parameters of the two solid materials. The design procedure of Moore *et al.* (1988) mentioned earlier is expressed directly in terms of $R_h$, so that the solutions for $R_h$ presented here are consistent.

*Effect of interface and Poisson's ratio*

To commence the study, consider how the interface condition and Poisson's ratio of the solid adjacent to the pipe $\nu_s$ and surrounding material $\nu_o$ influence stability. Solutions are shown in Fig. 3 for both bonded and smooth interface conditions, for $G_o/G_s = 0$ and 0.4, for $\nu_s = \nu_o$ equal to 0.25 and 0.48 and with $W/r = 0.3$. Also shown are a selection of critical mode numbers $n_{crit}$.

All curves show slope discontinuities associated with changes in critical mode $n_{crit}$. This is typical of linear buckling solutions based on harmonic analysis. Since values of $R_h = N_{crit}/N_{crit}^{\infty}$ are presented however, slope discontinuities arise from mode changes in the solutions for both the numerator $N_{crit}$ and the denominator $N_{crit}^{\infty}$.

Clearly, the effect of a zone of lower modulus material around the envelope of higher modulus soil is to reduce the restraint the soil provides to the cylinder. This can reduce critical hoop thrust, and correspond to a critical buckling mode which is substantially lower (i.e. buckle wavelength that is substantially longer) at a given value of $EI/G_s r^3$. For low values of $G_o/G_s$, the long wavelength mode $n_{crit} = 2$ is critical over a wide range of $EI/G_s r^3$ values.

For $G_o/G_s = 0.4$, the influence of the interface condition and Poisson's ratio is not very significant. It appears that the smooth interface condition yields a slightly more conservative result.

For $G_o/G_s = 0$, the influence of Poisson's ratio is still insignificant, but the interface condition now has a significant effect on the correction factor for low values of $EI/G_s r^3$ (i.e. less than 0.01).

While it is difficult to determine the precise nature of the interaction at the solid–structure interface of a buried structure during buckling collapse, it has been found through comparisons of theory with laboratory tests on soil supported pipes that the bonded interface condition is most appropriate (Moore, 1989). Poisson's ratio effect is not very significant, so a value of $\nu_s = \nu_o = 0.3$ has been employed with the bonded interface condition for the remainder of the study.

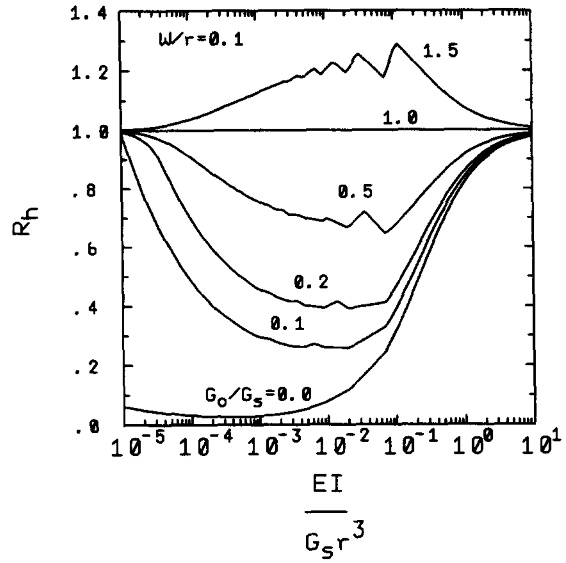


Fig. 4a. Correction factor $R_h$ for pipe in circular envelope—$W/r = 0.1$.

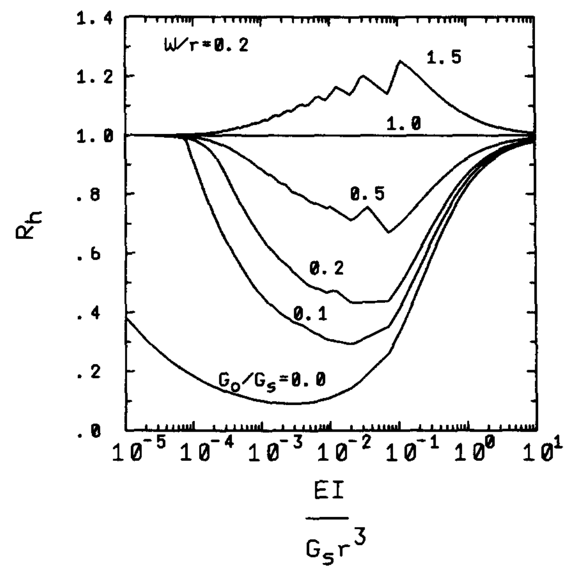


Fig. 4b. Correction factor $R_h$ for pipe in circular envelope—$W/r = 0.2$.

*General parametric solution*

Values of factor $R_h$ for a range of envelope geometries $W/r$, modulus ratios $G_o/G_s$ and normalized pipe stiffnesses $EI/G_s r^3$ are given in Figs 4a–d. Solutions for $R_h$ are obtained for $W/r$ of 0.1, 0.2, 0.5 and 1.0 and for shear modulus ratios $G_o/G_s$ of 0, 0.1, 0.2, 0.5, 1.0 and 1.5.

For $G_o/G_s = 1$ the solid is homogeneous so the correction factor $R_h$ is unity. If the solid immediately adjacent to the structure is stiffer than the material beyond it, then $R_h$ is less than unity. If the solid material adjacent to the structure is less stiff than the surrounding material, then $R_h$ is greater than unity. For $G_o/G_s = 0$, the ring of solid material is effectively surrounded by a material which provides no resistance to movement.

An examination of the results also reveals that the correction factor $R_h$ is only important for a limited range of $EI/G_s r^3$ values. The range limits are influenced by both the thickness of the envelope $W/r$ and the modulus ratio $G_o/G_s$. For this circular envelope and for typical flexible pipes where $10^{-5} < EI/G_s r^3 < 10^{-2}$, there is no significant reduction in buckling strength due to poor material beyond the stiff solid adjacent to the pipe provided $W/r$ is at

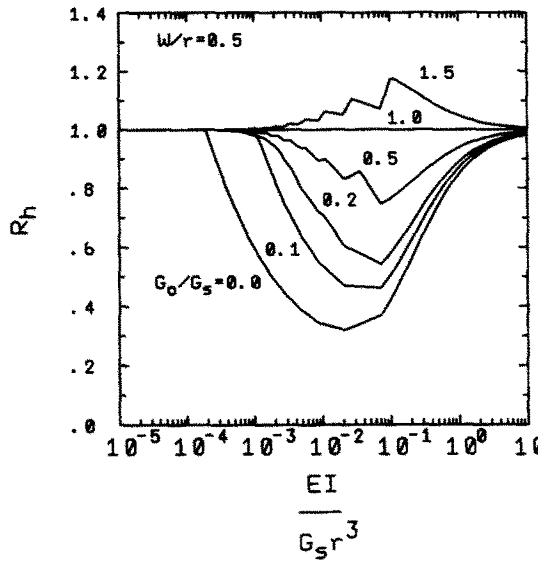


Fig. 4c. Correction factor $R_h$ for pipe in circular envelope—$W/r = 0.5$.

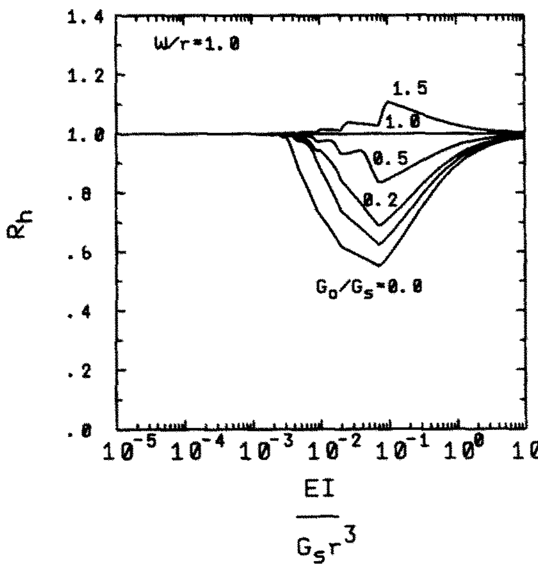


Fig. 4d. Correction factor $r_h$ for pipe in circular envelope—$W/r = 1$.

least equal to 1, Fig. 4d. This provides some support for empirical guidelines recommending the use of good quality backfill soil extending one half span beyond a flexible metal culvert (OHBDC, 1991). Structures could perhaps be designed with smaller backfill envelopes using Figs 4a–d or the closed form solution.

## FINITE ELEMENT SOLUTIONS FOR SHALLOW BURIED STRUCTURES

### *Introduction*

Flexible culverts with complex geometry are most easily analysed using the finite element method. The finite element analysis of critical thrust for buried flexible structures has been described previously in relation to pipes buried close to the ground surface (Moore, 1987) and noncircular pipes (Moore, 1988a,c). Briefly, the solid is modelled with six noded "linear strain" triangular elements and the boundaries of the solid are placed sufficiently far away so as to minimize boundary effects. The solid is represented as a linear elastic material. In the present study, the previous work is extended to include the effect of nonuniform modulus.

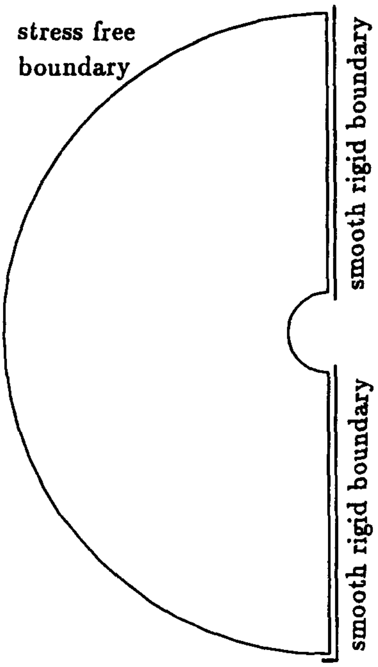


Fig. 5a. Outline of first finite element mesh.

Figure 5 shows details of two of the meshes used for the finite element study of pipe buckling. Figures 5a and 5b show a complete picture and a detail of the first mesh, and Figs 5c and 5d show a complete picture and detail of the second. The meshes are designed to extend a sufficient distance away from the cylinder so that the external boundary does not significantly affect the estimates of critical hoop thrust.

The structural elements as well as the continuum elements directly adjacent to the structure must be sufficiently fine so as to model the buckling deformations. For the mesh in Figs 5a and 5b, there are 64 structural elements used around the half-circle. This permits accurate solutions involving any buckling wavelength down to $\lambda \approx 2\pi r/20$, and solutions for thrust within 2% for $2\pi r/20 > \lambda > 2\pi r/30$. For the mesh in Figs 5c and 5d, there are 96 structural elements used around the half-circle. This permits accurate solutions involving any buckling wavelength down to $\lambda \approx 2\pi r/32$, and solutions for thrust within 2% for $2\pi r/32 > \lambda > 2\pi r/44$. Most solutions for critical thrust presented in this report are for high $\lambda$ (errors would be too small to detect on an $xy$-plot of results). However, some solutions at $EI/G_s r^3 < 0.0002$ are accurate to 2% only.

*Comparison with closed form solutions*

To demonstrate the effectiveness of the finite element solution, as well as providing evidence to verify the closed form buckling solution presented earlier in the paper, solutions for critical thrust have been calculated using the finite element and closed form solutions and compared in Fig. 6. The finite element solutions are obtained using the mesh shown in Fig. 5b with the inner three rings of solid elements forming the envelope of $G_s$ material, $W/r = 0.5$. The elastic material surrounding that ring is assigned shear modulus $G_o$, where $G_o/G_s$ alternately equals 0.1, 0.3 and 1. The solid and structure are modelled as bonded together at the interface. The values of critical thrust have all been normalized using

$$N_b = \frac{3EI}{r^2}\left(\left(\frac{G_s r^3}{EI}\right)^{\frac{2}{3}} + 1\right). \tag{25}$$

Clearly there is a close match between all of the closed form and finite element results. The only solutions which show some deviation are those for $G_o/G_s = 1$ where the finite element

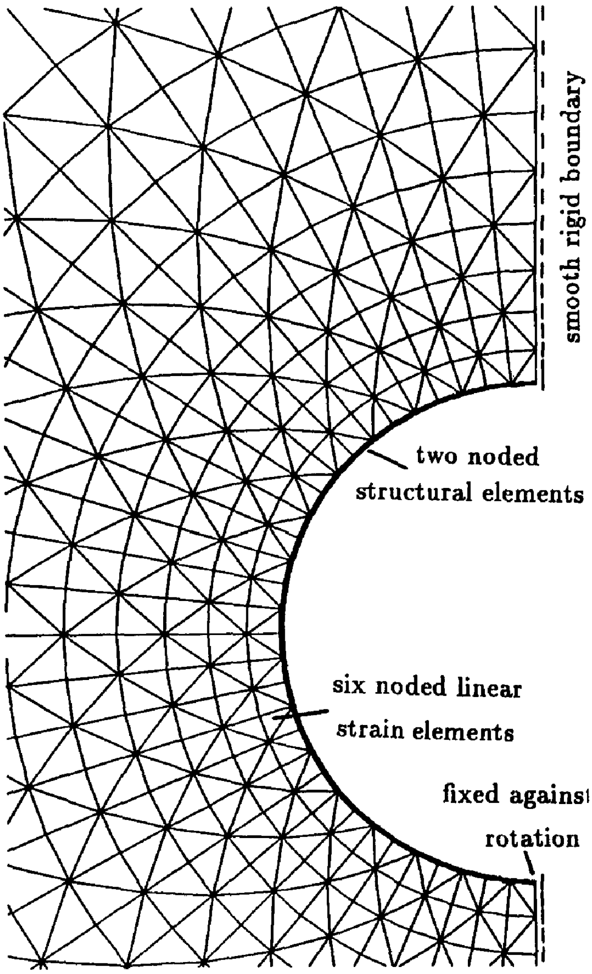


Fig. 5b. Detail of first finite element mesh.

solution is slightly less. The reason for this discrepancy is the nature of the external boundary of the finite element mesh which is assumed stress free. To confirm this, a closed form solution is shown as a dashed line on Fig. 6 for a uniform ring of elastic solid with stress free boundary at radial distance $8r$, the same location as the exterior boundary of the finite element mesh ($G_o/G_s$ is set to zero and $W/r$ to 8). The closed form solution then matches the finite element solution exactly. The solutions for $G_o/G_s < 1$ and $W/r = 0.5$ match better since the introduction of a zone of more compressible $G_o$ material reduces the boundary effect.

*Shape of the high modulus envelope*

The finite element analysis can be used to examine critical thrust for pipes buried in envelopes of various shapes. The mesh shown in Figs 5c and 5d has been used to examine pipe stability for square and rectangular envelopes. Figure 7 shows critical thrust $N_{crit}$ normalized using $N_b$ where $G_o/G_s = 0.1$. In addition to three cases of square envelopes, one case of rectangular zone and three cases of circular envelope (calculated using the closed form solution) have been included.

Examination of Fig. 7 reveals that:

- solutions for the mesh shown in Figs 5c and 5d are reasonable given the match of finite element and closed form solutions for extensive high modulus material, $W/r > 4$. The square zone solution is slightly higher for the range $0.08 < EI/G_s r^3 < 1$, since the exterior boundary of the finite element mesh is actually at $W/r = 4$ and is

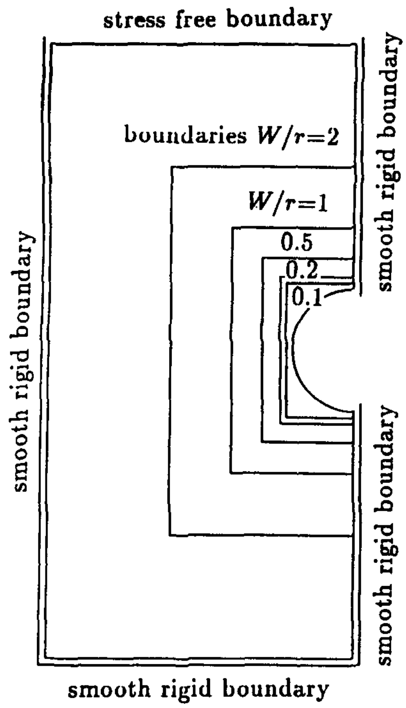


Fig. 5c. Outline of second finite element mesh.

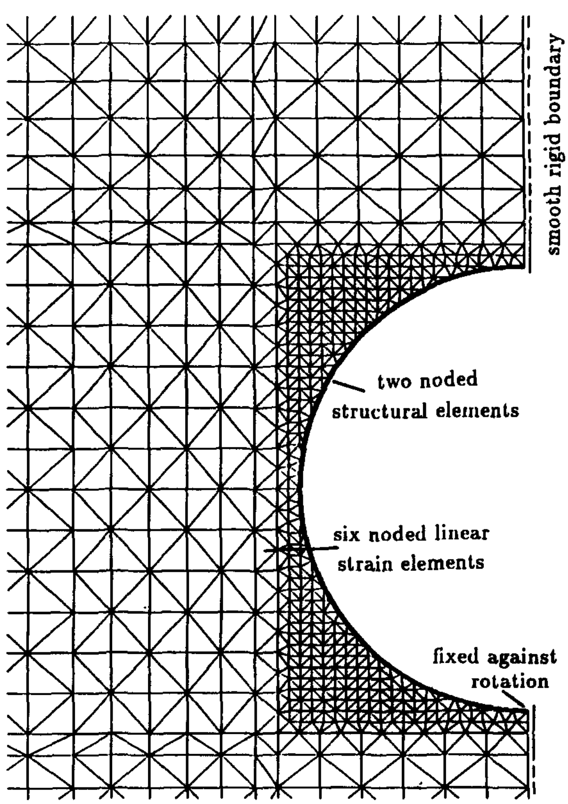


Fig. 5d. Detail of second finite element mesh.

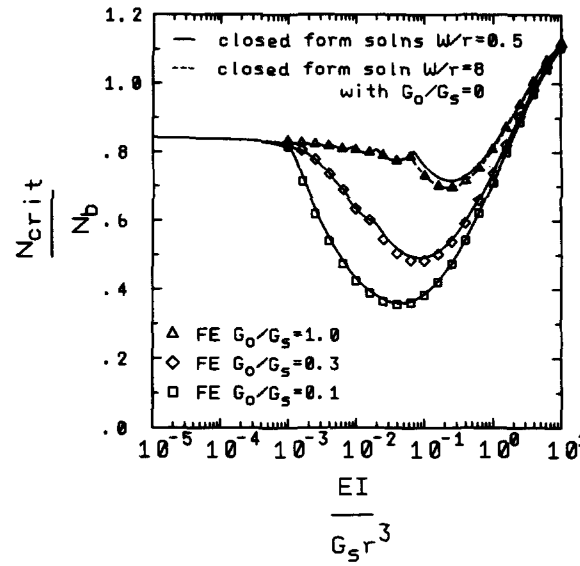


Fig. 6. Comparison of closed form and finite element solutions for normalized critical thrust.

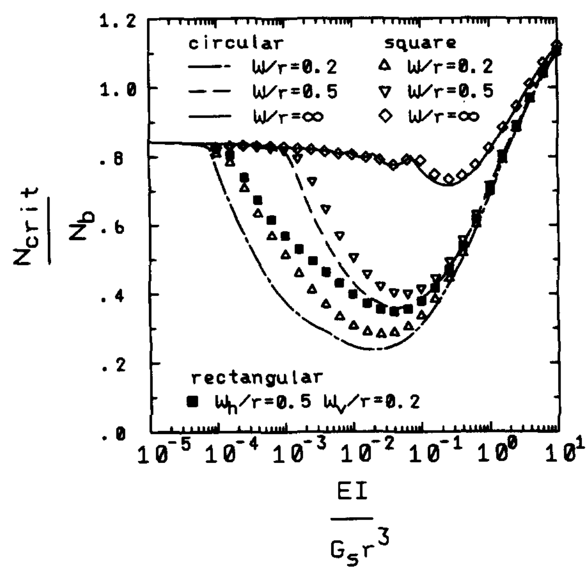


Fig. 7. Comparison of finite element solutions for normalized critical thrust showing the influence of high modulus solid geometry.

modelled as smooth and rigid. Any errors that occur for $W/r < 4$ will be even smaller since the more compressible $G_0$ material will reduce the boundary effect, and the errors are unnoticeable once smaller and more realistic values of $EI/G_s r^3$ are considered;

- as could be expected for $W/r$ equal to 0.2 and 0.5, the square zone incorporates more of the high modulus material so that it produces higher critical thrust than the annulus of $G_s$ material with the same minimum thickness $W/r$. For the cases shown, the maximum difference in critical thrust for the square and circular zone results is about 30%;
- for rectangular zone with thickness over the crown and under the invert $W_v/r$ equal to 0.2, and thickness adjacent to the structure $W_h/r$ equal to 0.5, the solution lies between the square envelope solutions with $W/r$ equal to those two values. As structural stiffness $EI/G_s r^3$ decreases, the wavelength of the buckling deformations decreases, so the behaviour at the springline becomes more dominant and the

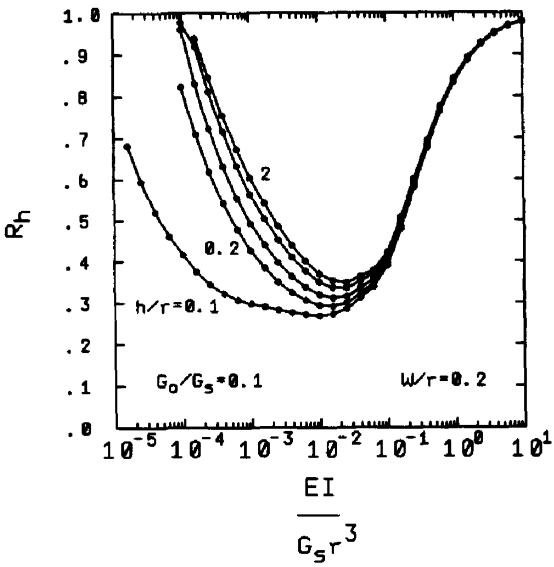


Fig. 8a. Correction factor $R_h$ for shallow buried pipe in square envelope—$W/r = 0.2$, $G_o/G_s = 0.1$.

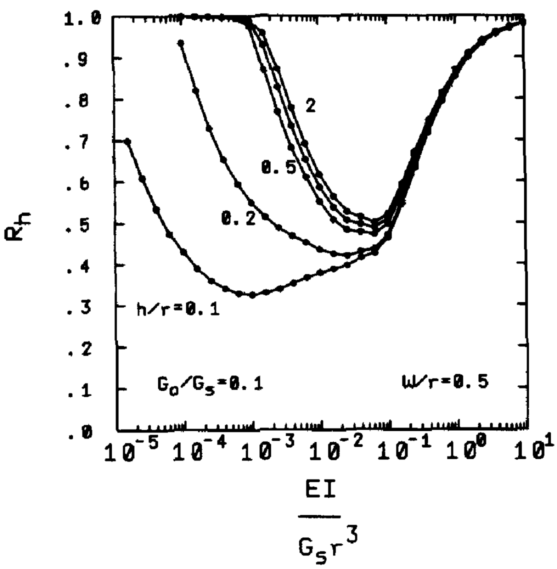


Fig. 8b. Correction factor $R_h$ for shallow buried pipe in square envelope—$W/r = 0.5$, $G_o/G_s = 0.1$.

solution approaches that of the square envelope with the lower amount of stiff $G_s$ material, $W/r = 0.2$.

Estimates of critical thrust for structures with different envelope geometries could be obtained using square zone or annular zone solutions for $R_h$, where $W/r$ is set to the minimum width of the $G_s$ material adjacent to the structure. These solutions would be somewhat, but not excessively, conservative.

*Burial depth*

One geometrical feature commonly present in large span buried flexible pipes or culverts, is shallow burial such that cover height above the crown $h$ may be only a fraction of the pipe radius. In order to examine this phenomenon, the mesh shown in Figs 5c and 5d has been used to obtain solutions for $R_h$ with material removed from the mesh above the pipe crown to model cover heights $h/r = 0.1$, 0.2, 0.5, 1 and 2:

- Fig. 8a shows solutions for $W/r = 0.2$ and $G_o/G_s = 0.1$;
- Fig. 8b shows solutions for $W/r = 0.5$ and $G_o/G_s = 0.1$;

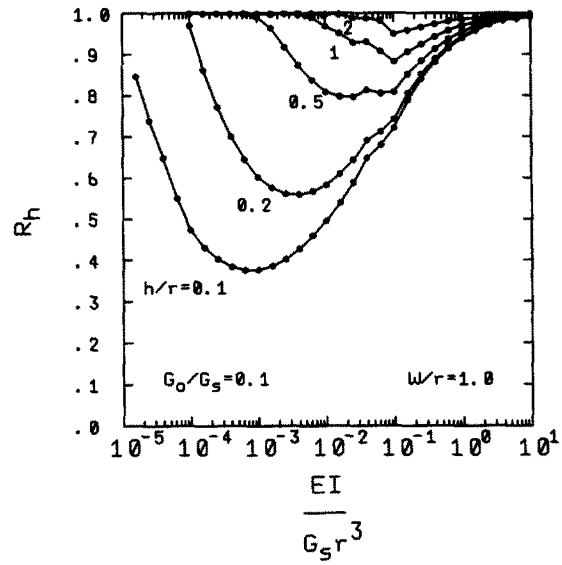


Fig. 8c. Correction factor $R_h$ for shallow buried pipe in square envelope—$W/r = 1$, $G_o/G_s = 0.1$.

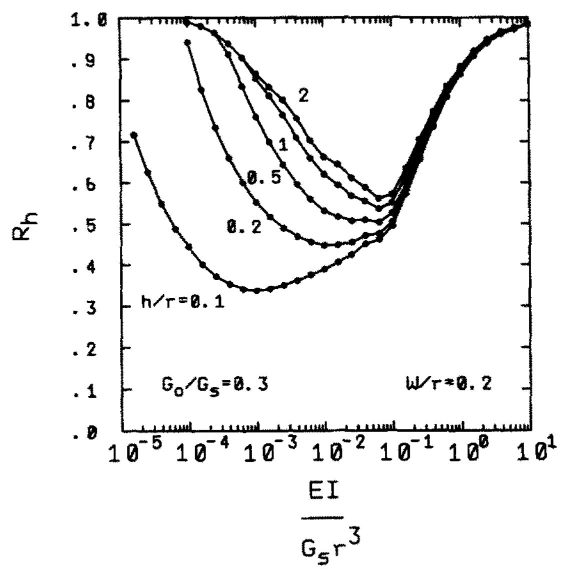


Fig. 9a. Correction factor $R_h$ for shallow buried pipe in square envelope—$W/r = 0.2$, $G_o/G_s = 0.3$.

- Fig. 8c shows solutions for $W/r = 1$ and $G_o/G_s = 0.1$;
- Fig. 9a shows solutions for $W/r = 0.2$ and $G_o/G_s = 0.3$;
- Fig. 9b shows solutions for $W/r = 0.5$ and $G_o/G_s = 0.3$;
- Fig. 9c shows solutions for $W/r = 1$ and $G_o/G_s = 0.3$;
- Fig. 10 shows solutions for $W/r = \infty$ or $G_o/G_s = 1$.

Using these solutions, the $R_h$ value for various burial depths $h/r$, envelope geometries $W/r$ and modular ratios $G_o/G_s$ can be estimated. An examination of the figures listed above reveals that there is a complex inter-relationship between critical thrust, burial depth, envelope geometry and modular ratio. Where burial depth $h$ is less than envelope width, then the stability is most affected by the cover condition, and decreases in $W/r$ or $G_o/G_s$ are not particularly significant. Alternatively, if dimension $W/r$ is substantially less than cover height $h/r$ then that dimension and the modular ratio $G_o/G_s$ are most important. Many problems in practice will have $h \approx W$, so that both the burial depth and the geometry of the zone of $G_s$ material influence the structural stability.

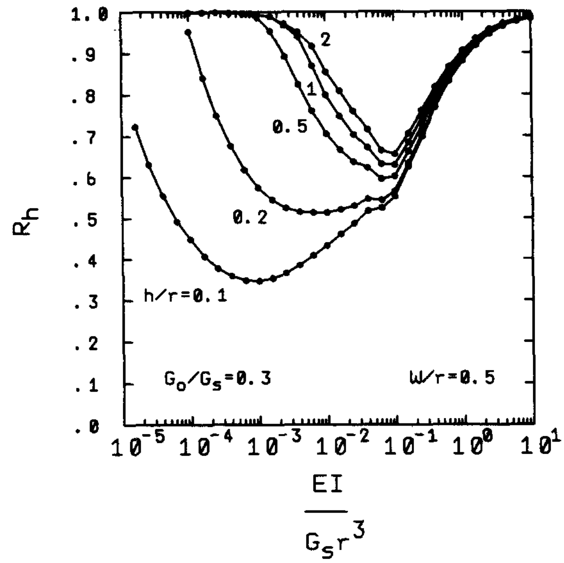


Fig. 9b. Correction factor $R_h$ for shallow buried pipe in square envelope—$W/r = 0.5$, $G_o/G_s = 0.3$.

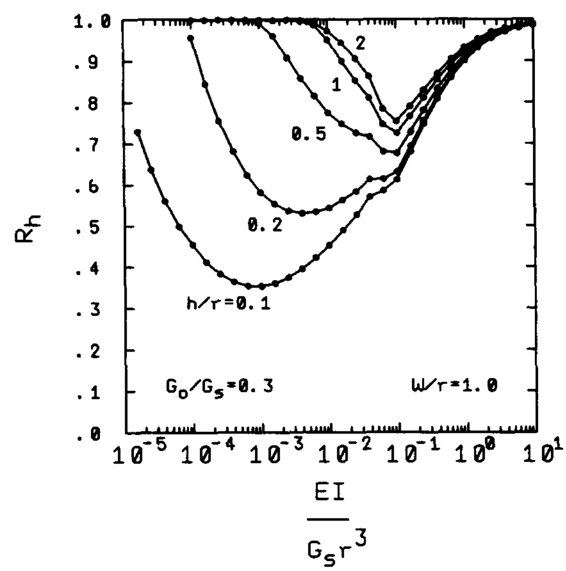


Fig. 9c. Correction factor $R_h$ for shallow buried pipe in square envelope—$W/r = 1$, $G_o/G_s = 0.3$.

*Cylinder shape*

The possibility of buckling failure often needs to be considered for culverts and pipes whose cross-section is elliptical or some other shape. Previous theoretical (Moore, 1987) and experimental (Moore, 1989) work on flexible structures buried in uniform elastic solid has demonstrated that the stability of an elliptical pipe can be estimated using solutions developed for circular pipe provided:

- the dimension $r$ used to normalize the pipe response is taken as the circumference of the pipe divided by $2\pi$, i.e. the radius of a circle of equal circumference;
- correction factors for shallow burial $R_h$ are normalized using half the pipe span $D_H/2$. Therefore, solutions presented in Fig. 10 can be employed provided $h/r$ is read as $2h/D_H$.

To investigate the behaviour of elliptical pipe buried in nonuniform elastic solid, Fig. 11 presents solutions for elliptical pipe buried in a rectangular zone of stiff solid with envelope thickness adjacent to the culvert and above and below, $W/r = 0.2$. The burial depths considered are $2h/D_H = 0.1$, 0.5 and 2. Also shown are the solutions for a circular culvert

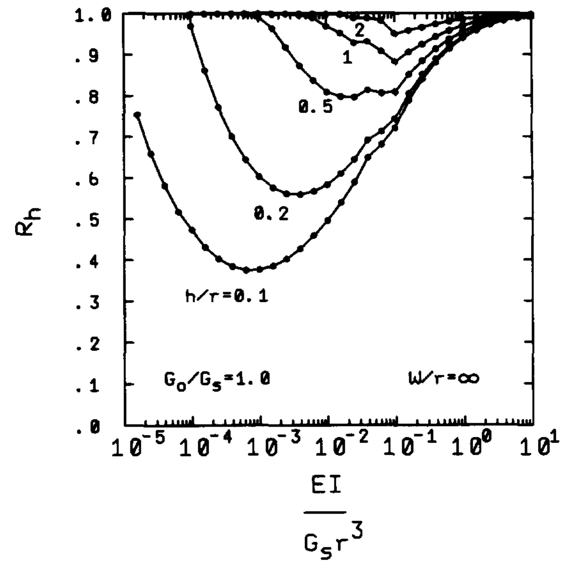


Fig. 10. Correction factor $R_h$ for shallow buried pipe in uniform solid.

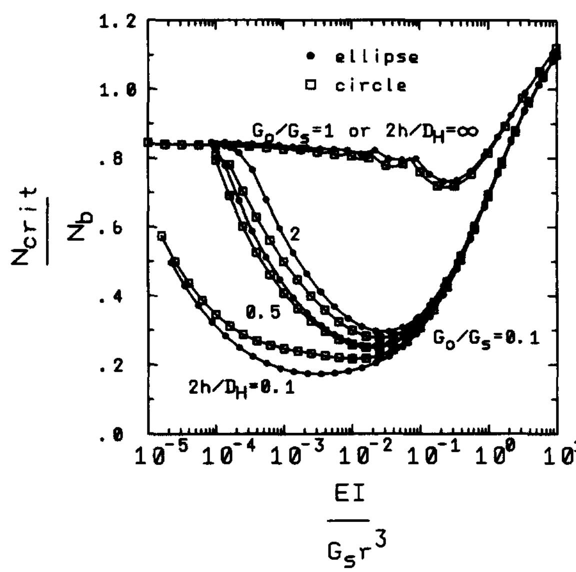


Fig. 11. Normalized thrust for shallow buried elliptical and circular pipes in nonuniform elastic solid—ellipse has $D_H/D_V = 2$.

with square zone of $G_s$ material with the same minimum thicknesses, as well as circular pipe and elliptical pipe solutions for uniform elastic ground. Modular ratio examined is $G_o/G_s = 0.1$, and the ellipse considered has span on rise ratio $D_H/D_V$ of 2.

Firstly, the elliptical and circular pipe solutions shown in Fig. 11 for deep burial in uniform material confirm that there is little significant difference in the critical thrust provided the solutions are normalized as described above. The comparisons for shallow buried structures in nonuniform solid show differences, but these are not particularly significant. For $2h/D_H = 0.1$ the elliptical pipe has lower normalized thrust around $EI/G_s r^3 = 10^{-3}$, but otherwise the normalized thrust for the circular pipe is similar or lower. Generally, it appears reasonable to use the circular pipe solutions to predict elliptical pipe response.

*Modulus that varies with depth*

To finish this study of elastically supported cylinder stability, the behaviour of a pipe buried in solid that has modulus which varies linearly with depth (a "Gibson" material) is considered,

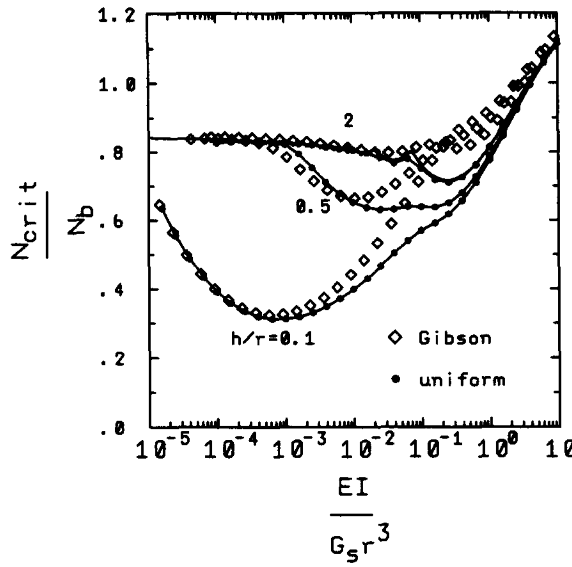


Fig. 12. Normalized thrust for shallow buried circular pipes in Gibson and uniform solids—for Gibson solid the modulus varies linearly with depth.

$$G_s = G_0 + gz \tag{26}$$

for surface modulus $G_0$, modulus gradient $g$ and depth $z$. Figure 12 shows normalized critical thrust for a range of pipe stiffnesses and three different burial depths, $h/r$ equals 0.1, 0.5 and 2. The solutions presented are for the particular case of $G_0 = gr$ and have been normalized using $G_s = G_0 + gh$, the shear modulus of the solid at the pipe crown.

Also shown in Fig. 12 are solutions for uniform solid $G_s = G_0 + gh$ and the same three burial depths. These match the Gibson solutions reasonably well, particularly in the important range $EI/G_s r^3 < 10^{-2}$. It appears that the uniform solid solutions can be used under these circumstances, where solid modulus at the crown is chosen to characterize the whole of the solid material. For real granular soil materials, "arching" around a shallow buried structure can be expected to reduce stresses and therefore modulus immediately above the crown. If this stress redistribution is considered in the analysis, the choice of modulus immediately above the crown seems likely to provide even more conservative solutions.

## CONCLUSIONS

The buckling strength of buried flexible structures has been examined using linear buckling theory for the shell structure with the soil represented as an elastic solid. In particular, the effect of nonuniform elastic modulus on the buckling strength has been considered. A closed form solution for a two-zone solid was presented, in addition to results obtained using the finite element method. Many of the solutions were presented using the correction factors $R_h$ which expresses the critical hoop thrust $N_{crit}$ for the pipe relative to critical thrust for a pipe buried in uniform solid.

Firstly, use of the closed form solution for pipe buried in a ring of stiff elastic solid revealed that Poisson's ratio for the solid does not significantly affect the correction factor $R_h$. Under some conditions, the interface condition between the solid and the structure was found to be important.

Using the finite element solution, the behaviour of pipes buried in square and rectangular envelopes of higher modulus solid were examined. It was found that in the absence of a solution for any particular envelope geometry, it is reasonable to use solutions developed for envelope with uniform thickness equal to the minimum for the real problem. Correction factor $R_h$ was also determined for various shallow buried nonuniform elastic solid configurations, and cases involving elliptical structures and solid modulus which varies linearly with distance from the free surface. A technique for normalizing the solutions was presented which permits elliptical pipe problems to be analysed using circular pipe solutions. It was also found that by using the solid modulus at the pipe crown, the behaviour of a pipe in a Gibson material could be estimated using the solutions for uniform modulus.

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